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Phase twisted modes and current reversals in a lattice model of waveguide arrays with nonlinear coupling

Michael Öster* and Magnus Johansson†

Department of Physics and Measurement Technology (IFM), Linköping University, S-581 83 Linköping, Sweden

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We consider a lattice model for waveguide arrays embedded in nonlinear Kerr media. Inclusion of nonlinear coupling results in many phenomena involving complex, phase-twisted, stationary modes. The norm (Poynting power) current of stable plane-wave solutions can be controlled in magnitude and direction, and may be reversed without symmetry-breaking perturbations. Also stable localized phase-twisted modes with zero current exist, which for particular parameter values may be compact and expressed analytically. The model also describes coupled Bose-Einstein condensates.

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Introduction. The use of nonlinear materials to create wave interference and demultiplexing/regeneration of high bit-rate channels in the optical domain is an integral part of high-speed optical communication. Whereas linear directional couplers have applications as simple optical switches [1], using nonlinear waveguide arrays for soliton-based (multiport) switching and beam steering is promising for future all-optical networks [2–4]. These ideas stem from the well-known properties of the lowest-order phenomenological model, the discrete nonlinear Schrödinger (DNLS) equation (e.g., [4,5]). The transmission characteristics of a two-waveguide device can be utilized for the construction of optical logical gates [6]. Incorporating nonlinear coupling between the waveguides, motivated, e.g., if the waveguides are assumed linear but embedded in a nonlinear Kerr material, multiguide arrays present features such as compact standing-wave solutions and mobility of strongly localized modes that may be used for multiport switching [7]. In this paper, we discuss *phase-twisted modes*, which due to the nonlinear interactions are found to exhibit a number of interesting properties, not present in the standard DNLS model.

Model. The model equation is [7]

$$\begin{aligned} i\dot{\Psi}_n = {} & Q_1\Psi_n + Q_2(\Psi_{n-1}+\Psi_{n+1}) + 2Q_3\Psi_n|\Psi_n|^2 \\ & + 2Q_4[2\Psi_n(|\Psi_{n-1}|^2+|\Psi_{n+1}|^2) + \Psi_n^*(\Psi_{n-1}^2+\Psi_{n+1}^2)] \\ & + 2Q_5[2|\Psi_n|^2(\Psi_{n-1}+\Psi_{n+1}) + \Psi_n^2(\Psi_{n-1}^*+\Psi_{n+1}^*) \\ & + \Psi_{n-1}|\Psi_{n-1}|^2 + \Psi_{n+1}|\Psi_{n+1}|^2], \end{aligned} \quad (1)$$

where $\Psi_n(z)$ is the complex amplitude of the electric field in the nth waveguide, the star denotes complex conjugation, and the dot differentiation with respect to z, the direction of propagation along the waveguides. The coupling constants Q_1–Q_5 depend on the overlap of the waveguide modes and the Kerr index of the surrounding medium (see explicit expressions in [7]). The same equation, with Q_4 argued to be negligible, was derived for the time evolution of Bose-Einstein condensates (BECs) in a periodic potential under a nonlinear tight-binding approximation [8], and may for $Q_3/2=Q_4=Q_5$ be found also modeling exciton-phonon coupling in α helices [9] or as a rotating-wave approximation to a Fermi-Pasta-Ulam (FPU) chain [10]. A transformation $\Psi_n(z)\mapsto a\Psi_n(bz)e^{-iQ_1 z}$ rescales the parameters in (1) according to $Q_1\mapsto 0$, $Q_2\mapsto Q_2/b\equiv K_2$ and $Q_j\mapsto Q_j|a|^2/b\equiv K_j$, $j=3, 4, 5$. Choosing $a^2=|Q_2/2Q_3|$ and $b=Q_2$ yields $K_2=1$ and $2K_3=\mathrm{sgn}(Q_2/Q_3)$. Since the transformation $\Psi_n\mapsto(-1)^n\Psi_n$ effectively changes the sign on Q_2 and Q_5, it will suffice to study the equation

$$\begin{aligned} i\dot{\Psi}_n = {} & \Psi_{n-1}+\Psi_{n+1}+\Psi_n|\Psi_n|^2 + 2K_4[2\Psi_n(|\Psi_{n-1}|^2+|\Psi_{n+1}|^2) \\ & + \Psi_n^*(\Psi_{n-1}^2+\Psi_{n+1}^2)] + 2K_5[2|\Psi_n|^2(\Psi_{n-1}+\Psi_{n+1}) \\ & + \Psi_n^2(\Psi_{n-1}^*+\Psi_{n+1}^*) + \Psi_{n-1}|\Psi_{n-1}|^2 + \Psi_{n+1}|\Psi_{n+1}|^2]. \end{aligned} \quad (2)$$

For general waveguide arrays, $Q_2>0$ and Q_j, $j=3, 4, 5$, has the sign of the Kerr index, which implies that we can, in (2), choose $K_4>0$ and K_5 to have the sign of the Kerr index. Making an estimate of the parameter values for an AlGaAs array of waveguides of the type considered in [7] with sizes of 10 μm and operated with a laser in the infrared ($\lambda\sim 1.5$ μm), we find $a^2=|Q_2/2Q_3|\sim 2$ kW. This can be compared, e.g., with the experimental value $|Q_2/2Q_3|=143$ W in [11] for a configuration with waveguides of a nonlinear material. Hence, the powers needed to operate an array with nonnegligible nonlinear coupling are accessible. Further, with a waveguide separation of the same order as the waveguide widths, the parameters K_4 and K_5 are of the order 0.1 (see [7]). In the BEC context, comparing with values calculated in [8] for ^{87}Rb atoms, we find $|Q_5/Q_2|\sim 10^{-2}-10^{-1}$, which is about two orders of magnitude larger than for the waveguide array. We also estimate $|K_5|=|Q_5/2Q_3|\sim 10^{-2}-10^{-1}$, where the sign is determined by the effective interatomic attraction (negative) or repulsion (positive).

Equation (2) has two conservation laws, which can be expressed in terms of discrete continuity equations. The first conserved quantity is the Hamiltonian $\mathcal{H}=\Sigma_n\mathcal{H}_n$, corresponding to invariance under translations in z, where the Hamiltonian density,

*Electronic address: micos@ifm.liu.se
†Electronic address: mjn@ifm.liu.se

$$\mathcal{H}_n = \Big[\Psi_n \Psi_{n+1}^* + \frac{1}{4}|\Psi_n|^4 + K_4(2|\Psi_n|^2|\Psi_{n+1}|^2 + \Psi_n^2 \Psi_{n+1}^{*2}) + 2K_5 \Psi_n \Psi_{n+1}(\Psi_n^{*2} + \Psi_{n+1}^{*2}) \Big] + \text{c.c.}, \quad (3)$$

satisfies $\dot{\mathcal{H}}_n + J_n^{(\mathcal{H})} - J_{n-1}^{(\mathcal{H})} = 0$, with the flux density

$$J_n^{(\mathcal{H})} = -2\,\mathrm{Re}\{\dot{\Psi}_{n+1}[\Psi_n^* + 2K_4\Psi_n^*(2\Psi_n\Psi_{n+1}^* + \Psi_n^*\Psi_{n+1}) + 2K_5\Psi_n^*(|\Psi_n|^2 + |\Psi_{n+1}|^2) + 2K_5\Psi_{n+1}^*(\Psi_n\Psi_{n+1}^* + \Psi_n^*\Psi_{n+1})]\}. \quad (4)$$

The second continuity equation is $\dot{\mathcal{N}}_n + J_n^{(\mathcal{N})} - J_{n-1}^{(\mathcal{N})} = 0$, with the norm density $\mathcal{N}_n = |\Psi_n|^2$ and current density

$$J_n^{(\mathcal{N})} = -2\,\mathrm{Im}\{[1 + 2K_4\Psi_n^*\Psi_{n+1} + 2K_5(|\Psi_n|^2 + |\Psi_{n+1}|^2)]\Psi_n^*\Psi_{n+1}\}. \quad (5)$$

The corresponding conserved quantity, the norm $\mathcal{N} = \Sigma_n \mathcal{N}_n$, is related to the invariance under the overall phase rotations of Ψ_n. Physically, this corresponds to the conservation of (Poynting) power along the waveguides, or, for BECs, to boson number conservation.

The intersite nonlinearities in (2) lead to a nontrivial norm current density and Hamiltonian flux density as compared to the DNLS equation ($K_4 = K_5 = 0$), and give rise to a range of new phenomena. Writing the complex amplitudes in action-angle variables $\Psi_n = \sqrt{\mathcal{N}_n} e^{-i\theta_n}$, with $\mathcal{N}_n$ and θ_n real, (5) simplifies to

$$J_n^{(\mathcal{N})} = -\frac{\partial \mathcal{H}_n}{\partial \phi_{n+1}} = 2\sqrt{\mathcal{N}_n \mathcal{N}_{n+1}} \sin\phi_{n+1} \times [1 + 4K_4\sqrt{\mathcal{N}_n\mathcal{N}_{n+1}}\cos\phi_{n+1} + 2K_5(\mathcal{N}_n + \mathcal{N}_{n+1})], \quad (6)$$

where $\phi_{n+1} \equiv \theta_{n+1} - \theta_n$. For *stationary* solutions, $\Psi_n(z) = \psi_n e^{-i\Lambda z}$, we also have $J_n^{(\mathcal{H})} = \Lambda J_n^{(\mathcal{N})}$ from (4). In general $J_n^{(\mathcal{N})} \neq 0$ for nontrivial phase twists ($\phi_n \neq 0, \pi$), so that for a solution to be stationary the net current flowing into site n from its two neighbors must be zero, i.e., $J_n^{(\mathcal{N})} = J_{n+1}^{(\mathcal{N})} \equiv J^{(\mathcal{N})}$. However, from (6) it follows that with intersite nonlinearities, we may have a zero norm current, $J_n^{(\mathcal{N})} = 0$, also for solutions with nontrivial phase twists, when

$$\cos\phi_{n+1} = -\frac{1 + 2K_5(\mathcal{N}_n + \mathcal{N}_{n+1})}{4K_4\sqrt{\mathcal{N}_n\mathcal{N}_{n+1}}}. \quad (7)$$

Equation (7) restricts the parameter values for the extra zero to appear since $|\cos\phi_{n+1}| \leq 1$. When $|\cos\phi_{n+1}| = 1$ phase-twisted solutions with $J_n^{(\mathcal{N})} = 0$ bifurcate from solutions without a twist ($\sin\phi_{n+1} = 0$), and these can generally be extended to a wider range of parameter values.

Constant-amplitude modes. A simple family of solutions to (2) is traveling plane waves of the form $\Psi_n(z) = \sqrt{\rho_0} e^{-i(\phi n + \Lambda z)}$, where the frequency is given by

$$\Lambda = \frac{\partial \mathcal{H}_n}{\partial \rho_0} = 2(1 + 8K_5\rho_0)\cos\phi + \rho_0 + 4K_4\rho_0(2\cos^2\phi + 1).$$

Their modulational stability is calculated (cf. [8,12]) by perturbing the solution, $\Psi_n(z) = [\sqrt{\rho_0} + u(z)e^{iqn} + v^*(z)e^{-iqn}]e^{-i(\phi n + \Lambda z)}$, and keeping only terms linear in u and v. This yields

$$i\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} a+b & c \\ -c & a-b \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \omega_\pm \begin{pmatrix} u \\ v \end{pmatrix}, \quad (8)$$

with $a = 2(1 + 8K_4\rho_0\cos\phi + 8K_5\rho_0)\sin\phi\sin q$, $b = -2\cos\phi + \rho_0 + 4K_4\rho_0[2\cos q(1 + \cos 2\phi) - \cos 2\phi] + 2(1 + 8K_5\rho_0)\cos\phi\cos q$, and $c = \rho_0 + 4K_4\rho_0(\cos 2\phi + 2\cos q) + 8K_5\rho_0\cos\phi(1 + \cos q)$. Plane waves are linearly stable if $\omega_\pm$ are real for all q. Explicitly, by writing

$$\omega_\pm = \frac{\partial J^{(\mathcal{N})}}{\partial \rho_0}\sin q \pm \sqrt{\frac{4}{m_{\mathcal{H}}}\sin^2\frac{q}{2}\left\{-2[(1 + 12K_5\rho_0)\cos\phi + 4K_4\rho_0(2\cos^2\phi + 1)]\sin^2\frac{q}{2} + \rho_0\frac{\partial\Lambda}{\partial\rho_0}\right\}}, \quad (9)$$

where we have introduced the (energetic) effective mass

$$\frac{1}{m_{\mathcal{H}}} = \frac{1}{\rho_0}\frac{\partial^2\mathcal{H}_n}{\partial\phi^2} = \frac{-1}{\rho_0}\frac{\partial J^{(\mathcal{N})}}{\partial\phi} = -2(1 + 4K_5\rho_0)\cos\phi - 8K_4\rho_0\cos 2\phi, \quad (10)$$

we see that the stability properties will be inverted when $m_{\mathcal{H}}$ changes sign, i.e., stable and unstable modulations in q will interchange. A point of marginal stability with $m_{\mathcal{H}}^{-1} = 0$ always separates stable and unstable solutions when $K_4 = 0$, while for $K_4 \neq 0$ solutions may be unstable on both sides. Note that in [8], the importance of the factor $m_{\mathcal{H}}^{-1}$ for the stability properties when $K_4 = 0$ was missed due to an approximation in the derivation of the eigenfrequencies. However, the nonequality of $m_{\mathcal{H}}$ and the dispersive effective mass m_Λ,

$$\frac{1}{m_\Lambda} = \frac{\partial}{\partial\rho_0}\frac{\partial^2\mathcal{H}_n}{\partial\phi^2} = \frac{\partial^2\Lambda}{\partial\phi^2} = -2(1 + 8K_5\rho_0)\cos\phi - 16K_4\rho_0\cos 2\phi, \quad (11)$$

where Λ corresponds to the chemical potential in the BEC context, was noted and connected to the dynamical properties of the condensates. Only for $K_4 = K_5 = 0$, the effective masses coincide and the criteria for instability reduces to the DNLS result $\partial^2\Lambda/\partial\phi^2 < 0$ [12].

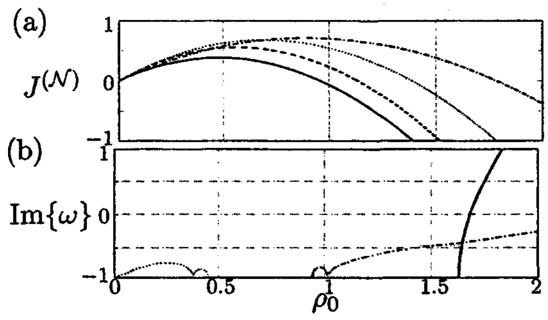

FIG. 1. (a) Norm current $J^{(\mathcal{N})}$, and (b) largest imaginary part of an eigenfrequency of (8) vs ρ_0, for $K_4=0.1$, $K_5=-0.2$, and different phase gradients: $\cos\phi=-0.6$ (solid), $\cos\phi=-0.2$ (dashed), $\cos\phi=0.2$ (dotted) and $\cos\phi=0.6$ (dash-dotted). For these parameter values, the region around the inversion point $J^{(\mathcal{N})}=0$ is stable for $-1<\cos\phi\lesssim 0.4$.

With the nonlinear dependence of the norm current (6), some control parameter can be used to govern the magnitude and, as opposed to the DNLS case ($K_4=K_5=0$), also *the direction* of the current for a given phase twist ϕ. With $K_4=0$ and $K_5<0$, the factor $1+4K_5\rho_0$ determines the sign of $J^{(\mathcal{N})}$. Hence, by varying the amplitude, the current can be *tuned to zero and its direction changed*. But, since the same factor appears in $m_{\mathcal{H}}^{-1}$ the stability properties will also be changed, i.e., the solution cannot be stable for both directions of the current around the inversion point. Including also the K_4 term, the zeros of $J^{(\mathcal{N})}$ and $m_{\mathcal{H}}^{-1}$ will depend on the phase gradient ϕ and in general not coincide. Numerical analysis of (9) for $K_4>0$ shows stable solutions, at least for all $K_4\lesssim 0.2$, around the current inversion point as exemplified in Fig. 1. For a given phase gradient $0<\phi<\pi$, the current will be positive (negative) for low (high) amplitudes for the parameter values in the figure. Hence, the power transfer across the waveguides can be controlled by changing the amplitude of the plane wave. The same mechanism also applies for a fixed amplitude and varying phase gradient, as shown in Fig. 2. For $K_4<0$, all solutions are found to be unstable near a current inversion point. Experimentally, the phase gradient can be controlled by launching the laser beam that excites the waveguides at an angle to the array (see, e.g., [13]).

A third way to control the direction of the current is by changing the configuration of the waveguides along the direction of propagation, i.e., making the coupling constants functions of z. In the BEC context this is equivalent to tuning the trapping potential as the condensates evolve. If the variation of the coupling constants is small compared to typical propagation wavelengths of the medium, we may still use (2) to describe the system. Further, the conservation of norm and the expression (5) for the norm current will still be valid, although the parameters depend on z. As an illustration (Fig. 3), we integrate (2) with $K_5=K_5(z)$ varying over a range with stable plane-wave solutions, keeping K_4 constant for simplicity, and calculate the norm current flowing through the lattice. Using periodic boundary conditions, i.e., modeling a circular array, implies a quantization of the phase gradient $\phi M\in 2\pi\mathbb{Z}$, where M is the number of sites. As $K_5(z)$ varies, the amplitude and the phase gradient are unchanged, within the size of the random perturbation, and only the frequency Λ is tuned. Thus, plane-wave solutions are robust to variation in the waveguide configuration along the direction of propagation, and the current can be reversed *without introducing any symmetry-breaking perturbation* in the system (2).

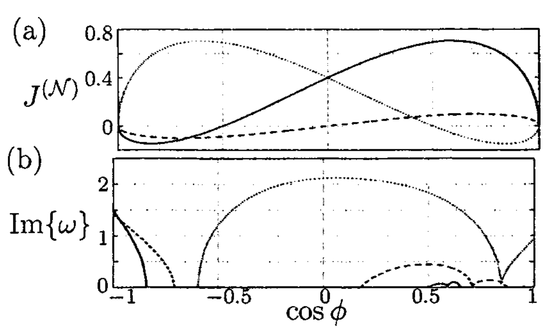

FIG. 2. (a) Norm current $J^{(\mathcal{N})}$, and (b) largest imaginary part of an eigenfrequency of (8) vs $\cos\phi$, for $K_4=0.1$, $K_5=-0.2$, and $\rho_0=1$ (solid); $K_4=0.1$, $K_5=-0.5$, and $\rho_0=0.5$ (dashed); $K_4=-0.1$, $K_5=-0.2$, and $\rho_0=1$ (dotted). Note the marginal stability when $m_{\mathcal{H}}^{-1}\propto\partial J^{(\mathcal{N})}/\partial\phi=0$.

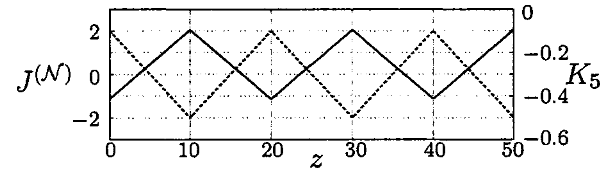

FIG. 3. Variation of the norm current $J^{(\mathcal{N})}$ (solid) with changing K_5 (dashed) as functions of z. The initial condition is a plane wave with $\rho_0=1$ and $\phi=-13\pi/25$ and a small random perturbation added. $K_4=0.1$, 50 sites and periodic boundary conditions were used.

Apart from the plane waves we may also consider other solutions with constant amplitude, e.g., solutions with a single nontrivial twist $\phi_{m+1}\equiv\phi$ satisfying (7) and $\phi_n=\begin{Bmatrix}0\\ \pi\end{Bmatrix}$ for $n\neq m+1$. The constraint (7) is necessary to have a stationary solution, since there is no norm current flowing through the lattice for solutions of this type. Inserting the ansatz in (2) it is required that $K_4=\mp K_5$ and $\Lambda=\pm 2(1+2K_5\rho_0)+\rho_0$, where the upper (lower) sign is for an unstaggered (staggered) background, $\phi_n=0$ ($\phi_n=\pi$). From (7) we have $\cos\phi=\pm[1+(4K_5\rho_0)^{-1}]$, which imposes $-2\leqslant 8K_5\rho_0\leqslant -1$. With these constraints inserted in (9), we see that the staggered background is always stable, while the unstaggered background is stable when $4K_5<-1$. However, numerics shows that an eigenmode localized around the twist always yields an instability, except at the bifurcation point $\phi=0$ where the solution with staggered background is stable for $K_5=-1/8\rho_0>-0.1146$.

Other types of stationary modes with nontrivial phase relations were recently found for the DNLS equation [14]. These solutions were constructed from two independent sublattices, each of constant amplitude and defined over odd and even sites, respectively, with the amplitudes out of phase on each sublattice and an arbitrary phase difference between the sublattices. Effectively the two sublattices are decoupled, but there is still a transfer of norm along the lattice, i.e., $J^{(\mathcal{N})}$

$\neq 0$. Such solutions can be found also for $K_4=0$ in our model (2).

Localized modes. Equation (2) also supports exact *compact* solutions [7], originating from an effective decoupling of parts of the lattice. From the linear-coupling terms and the last two terms in the K_5 part of (2), it follows that M-site compact solutions, with $\mathcal{N}_{m+j}=0$ for $j\leqslant -1$ and $j\geqslant M$, and $\mathcal{N}_{m+j}>0$ for $j=0,1,\ldots,M-1$, may exist if $1+2K_5\mathcal{N}_m=1+2K_5\mathcal{N}_{m+M-1}=0$. The simple real two-site compact solutions are given by $\mathcal{N}_m=\mathcal{N}_{m+1}=-1/2K_5$, $\phi_{m+1}=\begin{Bmatrix}0\\ \pi\end{Bmatrix}$ and $\Lambda=\mp 3-1/2K_5-3K_4/K_5$ [7]. Applying the constraint (7), which is necessary for localized solutions since $J_n^{(\mathcal{N})}=0$ at $n=\pm\infty$, we can also find *complex compact solutions*, with the same amplitude as above, the phase difference given by $\cos\phi_{m+1}=-K_5/2K_4$ and the frequency $\Lambda=-1/2K_5+K_5/K_4-K_4/K_5$. These complex solutions bifurcate from the real solutions at $K_5=-2|K_4|$, but it is not possible to continuously go from the symmetric to the antisymmetric real compact solution within the class of complex compact solutions, since excitations with $\cos\phi_{m+1}>0$ $(K_4>0)$ are separated from those with $\cos\phi_{m+1}<0$ $(K_4<0)$. A numerical investigation shows stability at both bifurcation points $(\phi_{m+1}=0,\pi)$, but only for $\cos\phi_{m+1}>0$ will we find stable compact solutions with a nontrivial twist for relatively small values of the parameters, $K_4\sim -K_5\lesssim 0.1$. Stable solutions for practically any phase twist $\cos\phi_{m+1}>0$ can be found as the magnitude of the parameters is decreased.

Also noncompact localized phase-twisted solutions may be investigated by taking the compact solutions as initial conditions in a Newton method following paths in parameter space (cf., e.g., [15]). For simplicity we study solutions with a symmetric profile, i.e., $\Psi_{m-n}=e^{i\phi_{m+1}}\Psi_{m+1+n}$ for all n and some site m. We assume also $\sin\phi_n=0$ for $n\neq m+1$, i.e., the solution has only a twist at the center. This restriction may be lifted, e.g., by applying the constraint (7) between each site, as well as by starting with other complex compact solutions, although finding analytic solutions with more sites excited is connected with an increasing algebraic complexity.

In Fig. 4 the results of the continuation for two cases, $\cos\phi_{m+1}$ positive and negative, are shown. Stable localized solutions with a single phase twist can be found in both cases. Note especially the stability around $K_4=0$, indicating that these types of solutions are also of relevance in the coupled BEC context. However, no solutions have been found for $K_5>0$, since the continuation could not be carried into this parameter regime. For normal DNLS $(K_4=K_5=0)$, localized stationary phase twisted modes cannot exist due to current conservation [16].

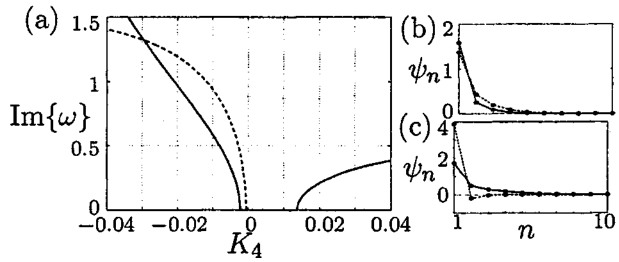

FIG. 4. (a) Largest imaginary part of an eigenfrequency vs K_4 from the linear stability analysis of localized modes with a single phase twist for $\cos\phi_1=-0.8333$ and $K_5=-0.1$ (solid), and for $\cos\phi_1=0.8333$ and $K_5=-0.05$ (dashed). The amplitude profiles for the two cases: (b) $[K_5=-0.1]$ $K_4=0$ (solid), and $K_4=0.04$ (dashed); (c) $[K_5=-0.05]$ $K_4=-0.04$ (solid) and $K_4=0.04$ (dashed). The solutions are compact for $K_4=-0.06$ and $K_4=0.03$, respectively. The second half of the amplitude profile is given by $\psi_{-n}=e^{i\phi_1}\psi_{n+1}$.

Conclusion. We have shown that taking into account nonlinear coupling in the DNLS model for waveguide arrays leads to a number of interesting phenomena, such as norm current reversal and stationary complex localized solutions that may be compact. Estimating the strength of the parameters in (1) indicates that effects of the nonlinear coupling can be observed experimentally, both for waveguide arrays and coupled BEC in the case $K_5>0$. However, the most interesting phenomena occur for $K_5<0$. Thus, a material with negative Kerr index or, alternatively, condensates with an effective interatomic attraction, of sufficient strength is needed.

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[1] A. Yariv, *Optical Electronics in Modern Communications*, 5th ed. (Oxford University Press, New York, 1997).
[2] H. S. Eisenberg *et al.*, Phys. Rev. Lett. **81**, 3383 (1998).
[3] O. Bang and P. D. Miller, Phys. Scr., T **67**, 26 (1996).
[4] Yu. S. Kivshar and G. P. Agrawal, *Optical Solitons—From Fibers to Photonic Crystals* (Academic Press, London, 2003).
[5] J. C. Eilbeck and M. Johansson, in *Localization and Energy Transfer in Nonlinear Systems*, edited by L. Vázquez *et al.* (World Scientific, Singapore, 2003), p. 44.
[6] S. M. Jensen, IEEE J. Quantum Electron. **18**, 1580 (1982).
[7] M. Öster, M. Johansson, and A. Eriksson, Phys. Rev. E **67**, 056606 (2003).
[8] A. Smerzi and A. Trombettoni, Chaos **13**, 766 (2003); Phys. Rev. A **68**, 023613 (2003); C. Menotti, A. Smerzi, and A. Trombettoni, New J. Phys. **5**, 112 (2003).
[9] K. Kundu, Phys. Rev. E **61**, 5839 (2000).
[10] Ch. Claude *et al.*, Phys. Rev. B **47**, 14 228 (1993).
[11] J. Meier *et al.*, Phys. Rev. Lett. **93**, 093903 (2004).
[12] Yu. S. Kivshar and M. Peyrard, Phys. Rev. A **46**, 3198 (1992).
[13] R. Morandotti *et al.*, Phys. Rev. Lett. **86**, 3296 (2001).
[14] L. Casetti and V. Penna, J. Low Temp. Phys. **126**, 455 (2002); M. Machholm *et al.*, Phys. Rev. A **69**, 043604 (2004); T. J. Alexander, A. A. Sukhorukov, and Yu. S. Kivshar, Phys. Rev. Lett. **93**, 063901 (2004).
[15] J. C. Eilbeck, P. S. Lomdahl, and A. C. Scott, Phys. Rev. B **30**, 4703 (1984).
[16] M. Johansson, Phys. Rev. E **66**, 048601 (2002).